

Randomly Amplified Discrete Langevin Systems

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Abstract

A discrete stochastic process involving random amplification with additive noise is studied analytically. If the non-negative random amplification factor b is such that $\langle b^\beta \rangle \geq 1$ where β is any positive non-integer, then the steady state probability density function for the process will have power law tails of the form $p(x) \sim 1/x^{\beta+1}$. This is a generalization of recent results for $0 < \beta < 2$ obtained by Takayasu et al. in Phys. Rev. Lett. **79**, 966 (1997). It is shown that the power spectrum of the time series x becomes Lorentzian, even when $1 < \beta < 2$, i.e., in case of divergent variance.

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Power law behavior of distribution function is widely observed in nature [1]. Recently, Takayasu et al. presented a new general mechanism leading to the power law distribution [2]. They analyzed a discrete stochastic process which involves random amplification together with additive external noise. They clarified necessary and sufficient conditions to realize steady power law fluctuation with divergent variance using a discrete version of linear Langevin equation expressed as

$$x(t+1) = b(t)x(t) + f(t), \quad (1)$$

where $f(t)$ represents a random additive noise and $b(t)$ is a non-negative stochastic coefficient. They derived the following time evolution equation for the characteristic function $Z(\rho, t)$ which is the Fourier transform of the probability density $p(x, t)$:

$$Z(\rho, t+1) = \int_0^\infty W(b)Z(b\rho, t)db\Phi(\rho), \quad (2)$$

where $W(b)$ is the probability density of $b(t)$ and $\Phi(\rho)$ is the characteristic function for $f(t)$. They showed that when $\langle b^\beta \rangle = 1$ holds for $0 < \beta < 2$, the second moment $\langle x^2(t) \rangle$ diverges as $t \rightarrow \infty$, but Eq. (2) has a unique steady and stable solution :

$$\lim_{t \rightarrow \infty} Z(\rho, t) \equiv Z(\rho) = 1 - \text{const} \times |\rho|^\beta + \dots, \quad (3)$$

which yields the power law tails in the steady probability density

$$\lim_{t \rightarrow \infty} p(x, t) \equiv p(x) \sim 1/x^{\beta+1}, \quad (4)$$

or equivalently, the cumulative distribution

$$P(\geq |x|) \sim 1/x^\beta. \quad (5)$$

They also made numerical simulations of Eq. (1) by employing a discrete exponential distribution for $W(b)$, and showed that the theoretical estimate of the relation between β and the parameters specifying $W(b)$ (Eq. (15) in [2]) nicely fits with the simulation “even out of the range of applicability, $\beta > 2$ ”. They state that “The reason for this lucky

coincidence is not clear”, although they point out at the same time that the power law distribution tails are a generic property of Eq. (1) [2].

In this Brief Report, the following two statements will be presented:

(A) Takayasu et al.’s theory can be straightforwardly extended for $\beta > 2$: If $\langle b^\beta \rangle = 1$ holds for a positive non-integer β , then there exists a unique steady and stable solution of Eq.(2)

$$Z(\rho) = \sum_{m=0}^n A_{2m} (-1)^{2m} \rho^{2m} - C |\rho|^\beta + O(\rho^{2n+2}), \quad (6)$$

where $2n$ is the largest even number that is smaller than β . This $Z(\rho)$ leads to $p(x) \sim 1/x^{\beta+1}$.

(B) When $\langle b^\beta \rangle = 1$ for a non-integer β between 1 and 2, the power spectral density (PSD) of $x(t)$ is Lorentzian increasing with the observation time T as

$$S(\omega, T) \sim \frac{2}{T} \frac{x_0^2}{\ln \langle b^2 \rangle} \frac{(1/\tau_1) \langle b^2 \rangle^T}{(1/\tau_1)^2 + \omega^2} \quad \text{for } T \gg 1, \quad (7)$$

where

$$x_0^2 \equiv \langle x^2(0) \rangle + \frac{\langle f^2 \rangle}{\langle b^2 \rangle - 1}, \quad (8)$$

$$\tau_1 = \frac{1}{\ln \langle b^2 \rangle + \ln[1/\langle b \rangle]}. \quad (9)$$

From the statement (A), “the coincidence” found in [2] is naturally understandable. To prove (A), we assume the following form for $Z(\rho)$:

$$Z(\rho) = \sum_{n=0}^{\infty} a_n \rho^n + |\rho|^\beta \sum_{n=0}^{\infty} c_n \rho^n, \quad a_0 \equiv 1, \quad (10)$$

and substitute it into Eq. (2) in the limit $t \rightarrow \infty$. If $\Phi(\rho)$ is an even function (i.e., the distribution function of $f(t)$ is symmetric as assumed in [2]), we can first prove that $a_1 = 0$ because $\langle b \rangle \neq 1$. Also, $c_1 = 0$ because $\langle b^{\beta+1} \rangle \neq 1$. Thanks to $a_{2m-1} = 0$ and $\langle b^{2m+1} \rangle \neq 1$, $a_{2m+1} = 0$ is derived. Similarly, $c_{2m-1} = 0$ and $\langle b^{\beta+2m+1} \rangle \neq 1$ yield $c_{2m+1} = 0$. We can thus prove that a_n and c_n in Eq. (10) vanish for all odd numbers n , i.e., Eq. (6) holds.

(Note that the n th moment $\langle x^n(t) \rangle$ with $n > \beta$ diverges not only for even number n but also for odd number which corresponds to the vanishing coefficient a_n .) Taking exactly the same procedures as in [2], we can prove that this solution is unique and stable. In case of $\beta > 2$, we have a finite variance but higher order moments, $\langle x^n(t) \rangle$ with $n > \beta$, diverge as $t \rightarrow \infty$.

To derive the probability density $p(x)$, we only need to assume that all k -th derivatives of $Z(\rho)$ satisfy the boundary condition

$$\lim_{\rho \rightarrow \pm\infty} d^k Z(\rho)/d\rho^k = 0. \quad (11)$$

Using Eq. (11), we can partially integrate the expression

$$p(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\rho} Z(\rho) d\rho \quad (12)$$

$[\beta] + 1$ times, where $[\beta]$ is the largest integer that is smaller than β . Thus we obtain the asymptotic expansion as

$$p(x) \sim |x|^{-(\beta+1)} \int_{-\infty}^{\infty} e^{-i\xi} |\xi|^{\beta-[\beta]-1} d\xi \sim |x|^{-(\beta+1)} \Gamma(\beta - [\beta]), \quad (13)$$

where Γ is the Gamma function.

To prove the statement (B), we note that the two-time correlation function is rigorously obtained from Eq. (1):

$$\phi(\tau, t) \equiv \langle x(t + \tau)x(t) \rangle = \langle x^2(t) \rangle \langle b \rangle^\tau, \quad (14)$$

where

$$\langle x^2(t) \rangle = \langle b^2 \rangle^t \langle x^2(0) \rangle + \frac{1 - \langle b^2 \rangle^t}{1 - \langle b^2 \rangle} \langle f^2 \rangle. \quad (15)$$

If $1 < \beta < 2$, we have a relation $0 < \langle b \rangle < 1 < \langle b^2 \rangle$ because the function $G(\gamma) \equiv \langle b^\gamma \rangle$ satisfies $G(0) = 1$ and $G''(\gamma) > 0$ [2]. Then ϕ increases with t , but decays with τ as $\sim e^{-\tau/\tau_0}$ for any fixed value of t (Debye-type relaxation), with the relaxation time

$$\tau_0 = \frac{1}{\ln[1/\langle b \rangle]}. \quad (16)$$

Since the correlation function depends on both τ and t , the Wiener-Khinchin relation cannot be used to obtain the PSD. Defining the PSD which depends on the observation time T as

$$\begin{aligned} S(\omega, T) &\equiv \left\langle \left| \int_0^T e^{i\omega t} x(t) dt \right|^2 \right\rangle / T \\ &= 2\text{Re} \left\{ \int_0^T d\tau \int_0^{T-\tau} dt e^{i\omega\tau} \langle x(t+\tau)x(t) \rangle \right\} / T, \end{aligned} \quad (17)$$

and using $\phi(\tau, t)$ obtained above, we arrive at the expression (7). The spectrum is $1/f^2$ -type for $f \gg 1/\tau_1$ and flat for $f \ll 1/\tau_1$. Equation (7) implies that the power increases exponentially with the observation time T , which corresponds to the divergent behavior of the variance $\langle x^2(t) \rangle$. (We have neglected the case $0 < \beta < 1$ where even the average of x diverges as $\langle x(t) \rangle = \langle b \rangle^t \langle x(0) \rangle$ because $\langle b \rangle > 1$.)

When $\beta > 2$, both $0 < \langle b \rangle < 1$ and $0 < \langle b^2 \rangle < 1$ hold and results are rather trivial:

$$\langle x^2 \rangle \equiv \lim_{t \rightarrow \infty} \langle x^2(t) \rangle = \frac{1}{1 - \langle b^2 \rangle} \langle f^2 \rangle, \quad (18)$$

$$\phi(\tau) \equiv \lim_{t \rightarrow \infty} \phi(\tau, t) = \langle x^2 \rangle \langle b \rangle^\tau, \quad (19)$$

$$S(\omega) \equiv \lim_{T \rightarrow \infty} S(\omega, T) = 2 \langle x^2 \rangle \frac{(1/\tau_0)}{(1/\tau_0)^2 + \omega^2}. \quad (20)$$

Thus, as far as the PSD is measured, we cannot observe any singular aspect, higher order singularities being hidden.

The stochastic process described by Eq. (1) generally leads to the power law behavior $p(x) \sim 1/x^{\beta+1}$, while it also yields a Lorentzian spectrum $S(\omega) \propto 1/[(1/\tau)^2 + \omega^2]$. A colored noise, or $1/f^\alpha$ fluctuation, whose PSD is proportional to $1/\omega^\alpha$ has attracted much attention since $1/f$ noise was discovered several decades ago[3]. Such power law behavior of the PSD is also observed widely in nature, and these two power laws, one in the probability density and the other in the PSD, are sometimes discussed together[2]. Therefore it is interesting to know whether an extremely long time scale τ can be involved in the present stochastic process. Because, in that case, the observation time T , which relates with the low frequency cut-off ω_0 as $\omega_0 = 2\pi/T$, cannot reach this time scale, then a $1/f^2$ fluctuation, namely $S(\omega) \sim 1/\omega^2$ (for $\omega \geq \omega_0$) is observed *practically*.

One can immediately see that the time constant τ_0 or τ_1 becomes large in very limited cases. First, the average of b should be close to unity, i.e., $\langle b \rangle = 1 - \epsilon$ with $0 < \epsilon \ll 1$. Then τ_0 becomes $\sim 1/\epsilon \gg 1$. Furthermore, in case of $\beta > 2$, we need $\langle b^2 \rangle$ smaller than unity, while in case of $1 < \beta < 2$, the condition $\langle b^2 \rangle = 1 + \delta$ with $0 < \delta \ll 1$ is necessary. In the latter case, we obtain $\tau_1 \sim 1/(\epsilon + \delta) \gg 1$. The exponential or Poisson distribution for $W(b)$ does not lead to such a long time constant. One example of large τ_1 is obtained by choosing $W(b)$ to be a narrowly peaked distribution having the average which is slightly smaller than unity and the second moment slightly larger than unity.

As pointed out in the above, it should be noted that a stochastic process whose stationary density function has power law tails will not necessarily exhibit the power law behavior in PSD.

References

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